

Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method

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Abstract

We introduce the *Mathematica* package **SummerTime** for arbitrary-precision computation of sums appearing in the results of DRA method. So far these results include the following families of the integrals: 3-loop onshell massless vertices, 3-loop onshell mass operator type integrals, 4-loop QED-type tadpoles, 4-loop massless propagators [1, 2, 3, 4]. The package can be used for high-precision numerical computation of the expansion coefficients of the integrals from the above families around arbitrary space-time dimension. In addition, this package can also be used for calculation of multiple zeta values, harmonic polylogarithms and other transcendental numbers expressed in terms of nested sums with factorized summand.

1. Introduction

Multiloop corrections within the Standard model and in general quantum field theory are of great interest now. In particular, this interest is connected with continuing search of New Physics both in collider experiments, such as LHC, and in high-precision spectroscopy measurement.

Probably, the most effective approach to the multiloop calculations is the differential equations technique. It was introduced long ago [5, 6, 7, 8] and has an impressive record of achievements. However, the differential equations do not work for one-scale integrals as their dependence on the scale is totally determined by dimensional considerations. On the other hand, the one-scale integrals are ubiquitous: they appear both in physical observables, like lepton anomalous magnetic moments or asymptotic $R(s)$ ratio, and as boundary conditions in differential equations for multiloop integrals.

Some time ago the DRA method of the calculation of the multiloop integrals has been introduced in Ref. [9]. This method is based on using the dimensional recurrence relations and analytical properties of the integrals as functions of the space-time dimensionality d . In a short time, the DRA method has been

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successfully applied to a wide range of physically interesting one-scale families of integrals [1, 2, 10, 11, 3, 4, 12, 13, 14].

The results of the DRA method are exact in d and have a form of nested sums with factorized dependence of the summand on the summation variables. Meanwhile, the applications require series expansion of the integrals in d near integer point, usually near $d = 4$. The most effective approach to obtaining analytical form of this expansion in terms of conventional transcendental numbers proved to be the one based on high-precision calculation of the expansion coefficients and subsequent use of the PSLQ algorithm. Many analytical results for the ϵ -expansions of the master integrals obtained in this way have been already published in the papers [1, 2, 10, 11, 3, 4, 12, 13, 14] near $d = 4$ (in Ref. [10] also near $d = 3$). However, the results of DRA method contain much more information than the published expansion coefficients. In fact, they are quite analogous to the representation in terms of hypergeometric functions, and totally determine expansion coefficients around any value of d . The only problem with these results is that, similar to the expansion of the hypergeometric functions, it may be quite difficult to express these coefficients in terms of conventional transcendental numbers, like multiple zeta values. Meanwhile, from time to time, there appears a necessity to calculate expansions of the master integrals either in different dimensionality or up to a higher order in ϵ .

The main goal of the present paper is to introduce a *Mathematica* package **SummerTime** for arbitrary-precision calculation of the expansion coefficients of DRA results around arbitrary value of d . This package gives high-energy community the full access to the results of the DRA method. In addition, the package contains procedures for high-precision evaluation of the multiple-zeta values and harmonic polylogarithms.

2. Solution of dimensional recurrence relation

Dimensional recurrence relations seem to appear for the first time in Ref. [15] and then rediscovered independently and systematically used in Ref. [16]. For the column-vector \mathbf{J} of the master integrals of a given sector (i.e., the integrals with a given set of denominators) the dimensional recurrence relation has the form

$$\mathbf{J}(\nu + 1) = \mathbb{C}(\nu)\mathbf{J}(\nu) + \mathbf{R}(\nu), \quad (1)$$

where inhomogeneous term $\mathbf{R}(\nu)$ contains master integrals from the subsectors, $\nu = d/2$. Let $\mathbb{S}(\nu)$ be a revertible matrix, satisfying the equation

$$\mathbb{S}(\nu) = \mathbb{S}(\nu + 1)\mathbb{C}(\nu). \quad (2)$$

Then the general solution of Eq. (1) can be written as follows:

$$\mathbf{J}(\nu) = \mathbb{S}^{-1}(\nu) \Sigma \mathbb{S}(\nu) \mathbb{C}^{-1}(\nu) \mathbf{R}(\nu). \quad (3)$$

Here Σ is the indefinite sum symbol with the property

$$(\Sigma F)(\nu + 1) - (\Sigma F)(\nu) = f(\nu). \quad (4)$$

If the function $F(\nu)$ decreases faster than $1/\nu$ when $\nu \rightarrow +\infty$ and/or $\nu \rightarrow -\infty$, one can write

$$\Sigma F(\nu) = \Sigma_{\geq} F(\nu) + \omega(z) \stackrel{\text{def}}{=} - \sum_{n=0}^{+\infty} F(\nu + n) + \omega(z) \quad (5)$$

and/or

$$\Sigma F(\nu) = \Sigma_{<} F(\nu) + \omega(z) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{-1} F(\nu + n) + \omega(z), \quad (6)$$

where $\omega(z) = \omega(\exp(2\pi i\nu))$ is arbitrary periodic function. The key feature of the DRA method is the determination of the function $\omega(z)$ from the analytical properties on the integrals as functions of ν .

Assuming that simpler master integrals in $\mathbf{R}(\nu)$ are also represented in a form (3), we naturally come to the problem of calculation of the nested sums of the form

$$\mathbb{J} = \mathbb{S}_K^{-1} \Sigma_{\tau_K} \mathbb{S}_K \dots \mathbb{S}_2^{-1} \Sigma_{\tau_2} \mathbb{S}_2 \mathbb{T}_{21} \mathbb{S}_1^{-1} \Sigma_{\tau_1} \mathbb{S}_1 \mathbb{T}_{10} \mathbb{S}_0^{-1}. \quad (7)$$

Here $\mathbb{T}_{i+1,i}(\nu)$ are rational rectangular matrices, $\tau_i \in \{\geq, <\}$, and $\mathbb{S}_i(\nu)$ is a hypergeometric matrix term, i.e. $\mathbb{S}_i^{-1}(\nu)\mathbb{S}_i(\nu+1)$ is a rational matrix. Explicitly, we have

$$\begin{aligned} \mathbb{J}(\nu) = \sum_{n_i \tau_i 0} \mathbb{S}_K^{-1}(\nu) \mathbb{S}_K(\nu + n_K) \dots (\mathbb{S}_2 \mathbb{T}_{21} \mathbb{S}_1^{-1})(\nu + n_K + \dots + n_2) \\ \times (\mathbb{S}_1 \mathbb{T}_{10} \mathbb{S}_0^{-1})(\nu + n_K + \dots + n_1). \end{aligned} \quad (8)$$

Here $n_i \tau_i 0$ denotes $n_i < 0$ or $n_i \geq 0$ depending on whether τ_i is equal to ‘<’ or ‘ \geq ’. Passing to the variables $k_i = n_i + n_{i+1} + \dots + n_K$, we obtain

$$\mathbb{J}(\nu) = \sum_{k_i \tau_i k_{i+1}} \mathbb{S}_K^{-1}(\nu) \mathbb{S}_K(\nu + k_K) \dots (\mathbb{S}_2 \mathbb{T}_{21} \mathbb{S}_1^{-1})(\nu + k_2) (\mathbb{S}_1 \mathbb{T}_{10} \mathbb{S}_0^{-1})(\nu + k_1). \quad (9)$$

In many physically interesting families of integrals each sector contains at most one master integral. Then all matrices are replaced by scalars and we come to the problem of evaluating the sums of the following form

$$J(\nu) = \sum_{k_i \tau_i k_{i+1}} F_K(\nu + k_K) \dots F_1(\nu + k_1), \quad (10)$$

where each F_i is a hypergeometric term, i.e. the ratio $F_i(\nu + 1)/F_i(\nu)$ is a rational function, and $k_{K+1} \equiv 0$. In particular such are all the families of integrals calculated within DRA method [1, 2, 10, 11, 3, 4, 12, 13, 14] except for Ref. [12] where the result for multimaster integral was obtained. The present version of **SummerTime** package is restricted to the calculation of the scalar sums (10).

3. Tree sums

Remarkable property of the sums (9) and (10) is that the dependence on the summation variables in the summand is factorized. If the limits of the sums were decoupled, the sum would be a product of one-fold sums. Nevertheless, even with the limits given by inequalities $k_i \tau_i k_{i+1}$, the sums in Eqs. (9) and (10) can be organized without nested loops as we explain below. But first we introduce the notion of tree sums and explain how sums of the form (9) or (10) can be rewritten in terms of those.

Let us define the following correspondence between a certain vertex-labeled directed graph and the solution of a set of inequalities. Namely, let us consider a directed graph with nodes marked by some expressions. Then the corresponding set of inequalities contains inequality $m \geq n$ for each edge $m \rightarrow n$. E.g., the set of integer triples (k_1, k_2, k_3) satisfying inequalities $(k_3 < 0) \& (k_2 \geq k_3) \& (k_1 < k_2)$ corresponds to the graph $\textcircled{0} \rightarrow \textcircled{k_3 + 1} \leftarrow \textcircled{k_2 + 1} \rightarrow \textcircled{k_1 + 2}$. All sums appearing in the DRA method can be labeled by path graphs. However, convergence rate analysis is easier for the sums which summation limits determined by the directed rooted tree graphs. Any path graph can be reduced to a set of directed tree graphs. E.g.,

$$\begin{aligned}
 \textcircled{0} \rightarrow \textcircled{k_3 + 1} \leftarrow \textcircled{k_2 + 1} \rightarrow \textcircled{k_1 + 2} &= \textcircled{0} \begin{array}{c} \nwarrow \textcircled{k_2} \nearrow \textcircled{k_1 + 1} \\ \searrow \textcircled{k_3 + 1} \end{array} + \textcircled{0} \rightarrow \textcircled{k_2 + 1} \begin{array}{c} \nwarrow \textcircled{k_3 + 1} \nearrow \textcircled{k_1 + 2} \end{array} \\
 &= \textcircled{0} \begin{array}{c} \nwarrow \textcircled{k_1 + 1} \nearrow \textcircled{k_2} \\ \searrow \textcircled{k_3 + 1} \end{array} + \textcircled{0} \begin{array}{c} \nwarrow \textcircled{k_1} \nearrow \textcircled{k_2 - 1} \\ \searrow \textcircled{k_3 + 1} \end{array} + \textcircled{0} \rightarrow \textcircled{k_2 + 1} \begin{array}{c} \nwarrow \textcircled{k_3 + 1} \nearrow \textcircled{k_1 + 2} \end{array} \\
 &= \textcircled{0} \begin{array}{c} \nwarrow \textcircled{-k_1 - 1} \nearrow \textcircled{k_2} \\ \searrow \textcircled{-k_3 - 1} \end{array} + \textcircled{0} \begin{array}{c} \nwarrow \textcircled{k_1} \nearrow \textcircled{k_2 - 1} \\ \searrow \textcircled{-k_3 - 1} \end{array} + \textcircled{0} \begin{array}{c} \nwarrow \textcircled{-k_2 - 1} \nearrow \textcircled{-k_3 - 1} \\ \searrow \textcircled{-k_1 - 2} \end{array} \quad (11)
 \end{aligned}$$

which corresponds to the identity

$$\begin{aligned}
 \sum_{\substack{k_3 < 0 \\ k_2 \geq k_3 \\ k_1 < k_2}} F(k_3, k_2, k_1) &= \sum_{\substack{0 \leq n_3 \\ 0 \leq n_2 \\ 0 \leq n_1}} F(-n_3 - 1, n_2, -n_1 - 1) \\
 + \sum_{\substack{0 \leq n_3 \\ 0 \leq n_1 \\ n_1 \leq n_2}} F(-n_3 - 1, n_2 + 1, n_1) &+ \sum_{\substack{0 \leq n_2 \\ n_2 \leq n_1 \\ n_2 \leq n_3}} F(-n_3 - 1, -n_2 - 1, -n_1 - 2). \quad (12)
 \end{aligned}$$

Note that if the summand on the left-hand side of Eq. (12) was a hypergeometric term with factorized dependence on the summation variables, the same is also true for the summands on right-hand side of the equation. The factorized form of the summand is a very important property which allows one to organize the calculation of the sums without nested loops.

In what follows we will use term ‘tree sum’ for the sums with limits determined by the directed rooted tree and with factorized dependence of the summand on the summation variables. It is very instructive to use the following alternative recursive definition of tree sums:

Definition 3.1. A *tree sum* is

- **Case 1:** Any one-fold sum $T(k) = \sum_{n=k}^{\infty} a(n)$
- **Case 2:** Any sum of the form $T(k) = \sum_{n=k}^{\infty} a(n) \prod_{i=1}^j T_i(n)$, where $T_i(n)$ are tree sums (j is a natural number).

3.1. Computation of tree sums

The numerical estimate of the tree sum can be obtained by replacing in the above definition the upper limit ∞ by some sufficiently large number N . A naive prescription would give the computational complexity $O(N^K)$ of such an estimate, where K is a ‘nestedness’ of the sum (the number of summation variables). In order to organize the computation much more effectively, we use recursive nature of the definition 3.1. We first note that the evaluation of the list of one-fold sums $\{T(0), T(1), \dots, T(N)\}$ can be done in one path with the computational complexity $O(N)$. Then, assuming that we already have such lists for each T_i , we can evaluate $T(k)$ from Case 2 of definition 3.1 also in $O(N)$. Therefore, the overall computational complexity of obtaining the estimate is $O(NK)$. We stress the difference between the computational complexities of the sums with factorized and nonfactorized summand.

Convergence acceleration. Once we eliminated the exponential dependence on K , we have to examine also the dependence of the complexity on the required precision P (P is a number of decimal digits). For exponentially converging sums (in what follows we call them *G-sums*) we have $N \sim P$ and the overall computational complexity is¹ $O(PK)$. However, for power-like behavior of the summand (we call such sums the *H-sums*) the number of terms in the naive computation depends exponentially on P . Therefore, we need to devise a suitable algorithm for convergence acceleration.

Let us consider a one-fold sum

$$T = \sum_{n=0}^{\infty} a(n), \quad a(n) \sim \frac{1}{n^{\alpha+1}} \text{ at large } n, \quad \alpha > 0. \quad (13)$$

Then in a naive prescription the required number of terms is $N \sim 10^{P/\alpha}$. Fortunately, it appears that we can use the convergence acceleration described in Ref. [17]. The general idea of the convergence acceleration is to replace the

¹We do not take into account the penalty of dealing with the P -digit numbers, which makes the complexity likely to be $O(P^2K)$.

original sum with the modified one converging to the same value but with faster convergence rate. Let the transformed sum have the form

$$T_1 = \sum_{n=0}^{\infty} a_1(n) \quad (14)$$

and

$$a_1(n) = (1 - f(n))a(n) + f(n+1)a(n+1), \quad (15)$$

where $f(n)$ is some function to be fixed and $a(-1) = 0$ by definition. Then the partial sum $T_1^{(N)}$ is expressed via $T^{(N)}$ as

$$T_1^{(N)} = \sum_{n=0}^N a_1(n) = T^{(N)} - f(0)a(0) + f(N+1)a(N+1). \quad (16)$$

The partial sums $T_1^{(N)}$ and $T^{(N)}$ both have the same limit when

$$f(0) = 0, \quad f(n)a(n) \xrightarrow{n \rightarrow \infty} 0. \quad (17)$$

The convergence rate is boosted when

$$a_1(n)/a(n) \xrightarrow{n \rightarrow \infty} 1 - f(n) + f(n+1)(1 + 1/n)^{-\alpha-1} \xrightarrow{n \rightarrow \infty} 0. \quad (18)$$

All these conditions are satisfied by $f(n) = n/\alpha$. Note that if we know N terms of the original sequence $a(n)$, we can reconstruct $N-1$ terms of the transformed sequence.

The leading asymptotics of the transformed term $a_1(n)$ depends on the next-to-leading term of $a(n)$. Let us assume that the asymptotic expansion of $a(n)$ has the form

$$a(n) \sim \sum_{k=1}^{\infty} \frac{c_{0,k}}{n^{\alpha+k}}. \quad (19)$$

Then the transformed sequence has the expansion

$$a_1(n) \sim \sum_{k=1}^{\infty} \frac{c_{1,k}}{n^{\alpha+1+k}}. \quad (20)$$

This allows one to apply the same acceleration technique recursively. If we have N terms of the sequence, we may apply the above acceleration technique $N-1$ times which gives an estimate $N \lg N \sim P$. Taking into account that computational complexity of the acceleration is $O(N^2)$, we have a rough estimate $O(P^2/\lg^2 P) \lesssim O(P^2)$ for the overall complexity of the power-like decaying sums. Note that the convergence acceleration is compatible with our algorithm of evaluation of the nested tree sums. The only complication comes from the fact that convergence acceleration leads to strong numerical compensations when calculating the terms of the accelerated sequence, therefore the working precision W should be taken higher than the required one, and the empirical rule is $W = \gamma P$, where $\gamma \approx 1.7$. Then, for K -fold sum we have $O(K\gamma^K P^2)$ complexity estimate.

Expansion in ϵ . Tree sums appearing in the DRA method depend on space-time dimensionality d . One is usually interested in their Laurent expansion near some point (typically, near $d = 4$). For a generic sum, depending on ϵ , the Laurent expansion does not always commute with the summation. However, the ϵ dependence of the tree sums appearing in the DRA method is very special: any summation variable n enters the summand only via combination $n + \epsilon$ or $n - \epsilon$. Therefore, the sums converge uniformly in ϵ which means that the Laurent expansion and the summation commute. Therefore, in order to calculate the former, one can first expand the summand and then make the summation. It is important that our convergence acceleration procedure is linear. Therefore we can apply it to formal power series in exactly the same way as to numbers.

4. Conventional transcendental numbers and functions as tree sums

The main purpose of the `SummerTime` package is to calculate sums which arise in the DRA method. However, it is remarkable that many conventional transcendental numbers and functions can be expressed and effectively evaluated as tree sums. Let us consider the Goncharov polylogarithms defined via

$$\text{Li}_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{0 < i_1 < i_2 < \dots < i_k} \frac{x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}}{i_1^{n_1} i_2^{n_2} \dots i_k^{n_k}} \quad (21)$$

To represent the above definition in the form of a tree sum, it suffices to rewrite it as

$$\text{Li}_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{0 \leq i_1 \leq \dots \leq i_k} \prod_{j=1}^k \frac{x_j^{i_j + j}}{(i_j + j)^{n_j}}. \quad (22)$$

Multiple zeta values

$$\zeta(n_k, \dots, n_1) = \sum_{i_k > \dots > i_1 > 0} \prod_{j=1}^k \frac{(\text{sgn } n_j)^{i_j}}{i_j^{|n_j|}} \quad (23)$$

are trivially related to the Goncharov polylogarithms:

$$\zeta(n_k, \dots, n_1) = \text{Li}_{|n_1|, \dots, |n_k|}(\text{sgn } n_1, \dots, \text{sgn } n_k), \quad (24)$$

and, therefore can be readily represented in the form of tree sums. Harmonic polylogarithms (HPL) are defined by (see, e.g., Refs. [18, 19])

$$H(\underbrace{0, \dots, 0}_n; x) = \frac{(\ln x)^n}{n!} \quad (25)$$

and

$$H(n, n_k, \dots, n_1; x) = \int_0^x f_n(t) H(n_k, \dots, n_1; x), \quad (26)$$

where $n \in \{0, \pm 1\}$ and

$$f_1(t) = \frac{1}{1-t}, \quad f_0(t) = \frac{1}{t}, \quad f_{-1}(t) = \frac{1}{1+t}. \quad (27)$$

It is standard to use nonzero integer n in the indices of H as a shortcut for sequence of $|n| - 1$ zeros appended by $\text{sgn } n$, e.g., $H(3, -2; x) = H(0, 0, 1, 0, -1; x)$. Since trailing zeros can be eliminated with the help of functional relations, see [19], we may consider only $H(n_k, \dots, n_1; x)$, where all n_1, \dots, n_k are nonzero integers. We have

$$H(n_k, \dots, n_1; x) = (\sigma_1 \sigma_2 \dots \sigma_k) \text{Li}_{|n_1|, \dots, |n_k|}(\sigma_1 \sigma_2, \sigma_2 \sigma_3, \dots, \sigma_k x), \quad (28)$$

where $\sigma_i = \text{sgn } n_i$. Therefore, HPLs also can be expressed via tree sums.

5. Application example

So far, the DRA method was applied to the following families of the integrals:

- To 3-loop onshell massless vertices [1];
- To 3-loop onshell mass operator type integrals [2];
- To 4-loop QED-type tadpoles [10];
- To 4-loop massless propagators [3, 4].
- Partly to 4-loop onshell mass operator type integrals [14].

The results of the DRA method for the first four families can be downloaded as `.zip` files from the package web page. Together with the `SummerTime` package, they allow one to calculate with arbitrary precision the expansion of the integrals near any value of d .

Let us present a minimal example of the application of `SummerTime` package. We assume that the content of the archive `M.zip` is put to a working directory of the *Mathematica* session. Then the following program calculates the expansion of $M_{6,2}$ integral near $d = 3$ up to ϵ^3 ($\epsilon = (3 - d)/2$) with the precision 1000 digits:

```
In[1]:= <<SummerTime'
SetDirectory[NotebookDirectory[]]
TriangleSumsSeries[<<"M62"/.d -> 3-2ϵ, {ϵ, 3}, 1000]
Out[1]:= - 126.330936333943790321081484798414734532015352412682120... × ϵ-2
+ 449.863309410101651375361946647344871810340547434784147... × ϵ-1
+ 7492.35397292225323538373824972066060688984257832532489815...
- 17762.4348499407483506220786909230708553370113393194388... × ϵ
- 148136.431403928982879168608861673380547613003055965761... × ϵ2
```


$$- 3048084.17207428077785717945708196781177068038009440576... \times \epsilon^3 \\ + O(\epsilon^4)$$

On the notebook with 2 cores Intel core i7 the run time of the above program was 12 min.

In order to showcase the usability of the package, we have calculated the expansions of all four-loop massless propagator integrals near $d = 3$ (the corresponding results for $d = 4$ can be found in Refs. [20, 3]). The results are presented in Appendix. The calculation provides a strong evidence of appearance of new transcendental numbers in addition to conventional ones: multiple zeta values and alternating harmonic sums.

6. Brief manual of SummerTime package

The `SummerTime` package can be downloaded from the site <http://www.inp.nsk.su/~lee/programs/SummerTime/> and installed according to the instruction included in a distribution archive. After the installation, the package can be loaded into *Mathematica* session by a command:

```
<< SummerTime'
```

The full list of functions provided by the package can be seen by `?SummerTime'`. Here we will consider only the basic ones.

6.1. Summation specifications

The `SummerTime` package supports two types of summation limit specifications. The first one, called *triangle*, is designed to handle the sums of form (10). This summation specification represents a list of variables with \pm signs, e.g.:

$$\{\pm n, \pm k, \dots\}. \quad (29)$$

Signs determine the direction of summation: $\{\dots, \pm n, k, \dots\}$ denotes $\pm n \leq k < \infty$ and $\{\dots, \pm n, -k, \dots\}$ denotes $\mp n \leq k < \infty$.

The second specification is used for tree sums, given by definition 3.1. It represents a tree, which is written in the following form:

$$\{\text{node}, \text{child}_1, \text{child}_2, \dots\}, \quad (30)$$

where children are also trees. If node doesn't have any children, it's simply written as:

$$\{\text{node}\}. \quad (31)$$

The node itself is a summation variable. The root of a tree is always an integer. For example, the specification for a tree sum with limits $0 \leq n \leq k < \infty$ can be written as $\{0, \{n, \{k\}\}\}$.

6.2. Summation functions

The **SummerTime** package defines following eight summation functions:

```
TreeSum[expr, specs, p]
TreeSumSeries[expr, specs, {e, o}, p]
TriangleSum[expr, specs, p]
TriangleSumSeries[expr, specs, {e, o}, p]
TreeSums[exprs, p]
TreeSumsSeries[exprs, {e, o}, p]
TriangleSums[exprs, p]
TriangleSumsSeries[exprs, {e, o}, p]
```

Here **expr** denotes a summand, **specs** is a summation specification (either tree or triangle), **exprs** is a list of pairs $\{\mathbf{expr}, \mathbf{specs}\}$, **e** is a small parameter ϵ , **o** is an expansion order with respect to ϵ , and **p** is required precision. Functions names are constructed in the following way: **Tree** (**Triangle**) means that function uses tree (triangle) summation specification, **Series** indicates that the expansion of sum in ϵ is performed, and **Sums** instead of **Sum** designates that several sums are calculated and their total is returned as a result. The advantage of **Sums** functions is that they perform computations in parallel.

All summation functions have an optional boolean argument **ProgressIndicator**, which denotes whether a progress bar must be shown during computations. Default value is **True**.

Sum functions also have an optional argument called **Information**. By this parameter, user can set up a string identifier, which will be shown on top of the progress bar. This option can be helpful for a sequence of calls of the summation functions: using it, user can understand which particular sum is currently being calculated.

Parallel functions (**Sums**) also have an optional boolean argument **Parallel**, which toggles, whether computation must be parallelized. Default value is **True**.

6.3. Other functions

Currently, the **SummerTime** package contains also implementation of multiple-zeta values and harmonic and Goncharov polylogarithms. By default these functions are disabled and in order to use them, one needs to set up a value of special variable **\$Options** before loading the package:

```
SummerTime`Package`$Options = {"MZV" -> True, "HPL" -> True,
                                "MPL" -> True};

<< SummerTime`
```

Multiple zeta values, harmonic and Goncharov polylogarithms are available via

$\text{MZV}[\mathbf{m1}, \dots, \mathbf{mk}]$	$\zeta(m_1, \dots, m_k)$
$\text{HPL}[\{\mathbf{m1}, \dots, \mathbf{mk}\}, \mathbf{x}]$	$H(m_1, \dots, m_k; x)$
$\text{MPL}[\{\mathbf{m1}, \mathbf{x1}\}, \dots, \{\mathbf{mk}, \mathbf{xk}\}, \mathbf{x}]$	$\text{Li}_{m_1, \dots, m_k}(x_1, \dots, x_k; x)$

Numerical values of these functions can be obtained via standard *Mathematica* function **N**.

7. Conclusion

In the present paper we have introduced a *Mathematica* package `SummerTime` for calculation of the multiloop integrals obtained within DRA method. The main purpose of the package is an arbitrary-precision calculation of the expansion coefficients of the integrals around any value of d . In addition, the package contains convenience tools for the calculation of various transcendental constants and functions: multiple zeta value, harmonic polylogarithms and Goncharov polylogarithms.

In the future, we plan to extend the package to the computation of the matrix sums. The difficulties connected with the convergence acceleration of the matrix sums seem to be possible to overcome using the approach of Ref. [21]. Such an extension of the package is highly desirable given the families of integrals of actual interest for the present moment.

Acknowledgments. This work was supported by RFBR grant No. 15-02-07893.

Appendix A. Expansion of massless propagator-type master integrals

Here we present the results for the expansions of the massless four-loop propagator-type master integrals around $d = 3$ ($\epsilon = (3 - d)/2$). It turns out [22] that these expansions may be of some interest for the applications.

We fix normalization by dividing all expansions with

$$M_{01}(3 - 2\epsilon) = \frac{\Gamma(\frac{1}{2} - \epsilon)^5 \Gamma(4\epsilon - 1)}{\Gamma(\frac{5}{2} - 5\epsilon)} \quad (\text{A.1})$$

All results have been obtained by numerically calculating the expansions of the results of Ref. [4] and then using *Mathematica* built-in function `FindIntegerNullVector` with a set including non-alternating and alternating harmonic sums. For some coefficients this approach failed which strongly indicates appearance of new constants in the expansions. We added the unresolved coefficients to the transcendental basis and used the extended basis for subsequent integrals. In the end we have expressed all expansions in terms of standard transcendental numbers

$$\zeta_n = \sum_{i=1}^{\infty} \frac{1}{i^n}, \quad a_n = \sum_{k=1}^{\infty} \frac{1}{2^k k^n}, \quad s_6 = \sum_{m=1}^{\infty} \sum_{k=1}^m \frac{(-1)^{m+k}}{m^5 k} \quad (\text{A.2})$$

and the following ten unresolved coefficients:

$$M_{21}^{(1)} = -262.1990984105955060153722678925879686999467800030 \dots, \quad (\text{A.3})$$

$$M_{21}^{(2)} = 2873.2506195273103871316630429729277291187753911438 \dots, \quad (\text{A.4})$$

$$M_{21}^{(3)} = -20850.77252988625214437845006030730940289184143944 \dots, \quad (\text{A.5})$$

$$M_{21}^{(4)} = 123806.28243923947828962358530545969884163429504554 \dots, \quad (\text{A.6})$$

$$M_{22}^{(2)} = 1915.2828601394850124013600200661647497224250709008 \dots, \quad (\text{A.7})$$

$$M_{22}^{(3)} = -9230.990439724705220633706650782234978383599409697 \dots, \quad (\text{A.8})$$

$$M_{22}^{(4)} = 24179.184087947556033170242057981283994228407821870 \dots, \quad (\text{A.9})$$

$$M_{35}^{(5)} = 20971.632987103828923594052378653898688768972653284 \dots, \quad (\text{A.10})$$

$$M_{41}^{(3)} = 30385.287535356282840631913693445372174956734072988 \dots, \quad (\text{A.11})$$

$$M_{41}^{(4)} = -157029.840084536053310520481290249759485412480424 \dots \quad (\text{A.12})$$

The notation of these constants is as follows: $M_A^{(n)}$ denotes the expansion coefficient of $M_A(3-2\epsilon)/M_{01}(3-2\epsilon)$ in front of ϵ^n .

We present results only for the integrals which can not be expressed in terms of Γ -functions. Where it is possible, we pull out rational factor making the expansion uniform in transcendental weight.

$$\frac{M_{21}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = M_{21}^{(1)}\epsilon + M_{21}^{(2)}\epsilon^2 + M_{21}^{(3)}\epsilon^3 + M_{21}^{(4)}\epsilon^4 + O(\epsilon^5) \quad (\text{A.13})$$

$$\frac{M_{22}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = M_{22}^{(1)}\epsilon + M_{22}^{(2)}\epsilon^2 + M_{22}^{(3)}\epsilon^3 + M_{22}^{(4)}\epsilon^4 + O(\epsilon^5) \quad (\text{A.14})$$

$$\begin{aligned} \frac{M_{23}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = & -\frac{\pi^2 2^{4\epsilon} (4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{6\epsilon-1} \left\{ 1 + 20\zeta_2\epsilon^2 + 242\zeta_3\epsilon^3 \right. \\ & \left. + 2487\zeta_4\epsilon^4 + (4840\zeta_2\zeta_3 + 18054\zeta_5)\epsilon^5 + (29282\zeta_3^2 + 231135\zeta_6)\epsilon^6 + O(\epsilon^7) \right\} \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \frac{M_{24}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = & \pi^4 2^{8\epsilon} \epsilon (4\epsilon-1)(10\epsilon-3)(10\epsilon-1) \left\{ 1 + 48\zeta_2\epsilon^2 + 284\zeta_3\epsilon^3 + 5706\zeta_4\epsilon^4 \right. \\ & \left. + (13632\zeta_2\zeta_3 + 19356\zeta_5)\epsilon^5 + (40328\zeta_3^2 + 496040\zeta_6)\epsilon^6 + O(\epsilon^7) \right\} \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} \frac{M_{25}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = & -\frac{\pi^2 2^{4\epsilon-1} (4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{6\epsilon+1} \left\{ 1 - 4\zeta_2\epsilon^2 + 74\zeta_3\epsilon^3 \right. \\ & \left. - 213\zeta_4\epsilon^4 + (1686\zeta_5 - 296\zeta_2\zeta_3)\epsilon^5 + (2738\zeta_3^2 - 5367\zeta_6)\epsilon^6 + O(\epsilon^7) \right\} \end{aligned} \quad (\text{A.17})$$

$$\frac{M_{26}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = -\frac{(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{6\epsilon+1} \left\{ 3\zeta_2 + (42\zeta_3 - 12a_1\zeta_2)\epsilon \right.$$

$$\begin{aligned}
& + (-24a_1^2\zeta_2 + 8a_1^4 + 192a_4 + 81\zeta_4)\epsilon^2 + \left(32a_1^3\zeta_2 - 324a_1\zeta_4 - \frac{32a_1^5}{5} + 768a_5 + 42\zeta_2\zeta_3 \right. \\
& \left. + 1023\zeta_5 \right) \epsilon^3 + \left(-32a_1^4\zeta_2 + 648a_1^2\zeta_4 - 168a_1\zeta_2\zeta_3 + \frac{64a_1^6}{15} + 3072a_6 + 1086\zeta_3^2 \right. \\
& \left. + \frac{23967\zeta_6}{4} - 2112s_6 \right) \epsilon^4 + O(\epsilon^5) \Big\} \quad (\text{A.18})
\end{aligned}$$

$$\begin{aligned}
\frac{M_{27}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= -\frac{(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{6\epsilon-1} \left\{ 6\zeta_2 + (48a_1\zeta_2 - 42\zeta_3)\epsilon + (96a_1^2\zeta_2 \right. \\
& + 16a_1^4 + 384a_4 + 297\zeta_4)\epsilon^2 + \left(256a_1^3\zeta_2 + 2376a_1\zeta_4 + \frac{128a_1^5}{5} - 3072a_5 + 1704\zeta_2\zeta_3 \right. \\
& - 1302\zeta_5 \Big) \epsilon^3 + \left(512a_1^4\zeta_2 + 9504a_1^2\zeta_4 + 13632a_1\zeta_2\zeta_3 + \frac{512a_1^6}{15} + 24576a_6 - 3948\zeta_3^2 \right. \\
& \left. \left. + 18891\zeta_6 - 5376s_6 \right) \epsilon^4 + O(\epsilon^5) \right\} \quad (\text{A.19})
\end{aligned}$$

$$\begin{aligned}
\frac{M_{32}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= \pi^2(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)\epsilon \left\{ 6\zeta_2 + 84\zeta_3\epsilon + (-96a_1^2\zeta_2 \right. \\
& + 336a_1\zeta_3 + 16a_1^4 + 384a_4 + 702\zeta_4)\epsilon^2 + \left(-256a_1^3\zeta_2 + 672a_1^2\zeta_3 + \frac{256a_1^5}{5} + 1536a_4a_1 \right. \\
& + 1536a_5 + 4464\zeta_2\zeta_3 + 2046\zeta_5 \Big) \epsilon^3 + \left(192a_1^4\zeta_2 + 896a_1^3\zeta_3 - 8640a_1^2\zeta_4 + 12096a_1\zeta_2\zeta_3 \right. \\
& + 8184a_1\zeta_5 + 13824a_4\zeta_2 + \frac{256a_1^6}{3} + 3072a_4a_1^2 + 6144a_5a_1 + 6144a_6 + 21156\zeta_3^2 + 65691\zeta_6 \\
& \left. \left. - 4224s_6 \right) \epsilon^4 + O(\epsilon^5) \right\} \quad (\text{A.20})
\end{aligned}$$

$$\begin{aligned}
\frac{M_{33}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= -\frac{\pi^2(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{(6\epsilon+1)^2} \left\{ 1 + (4a_1 + 14)\epsilon + (8a_1^2 + 56a_1 \right. \\
& + 60\zeta_2)\epsilon^2 + \left(624a_1\zeta_2 + \frac{32a_1^3}{3} + 112a_1^2 + 456\zeta_2 - 278\zeta_3 \right) \epsilon^3 + \left(2784a_1^2\zeta_2 + 4128a_1\zeta_2 \right. \\
& - 1112a_1\zeta_3 + \frac{416a_1^4}{3} + \frac{448a_1^3}{3} + 3072a_4 - 1204\zeta_3 + 4299\zeta_4 \Big) \epsilon^4 + \left(8832a_1^3\zeta_2 \right. \\
& + 17472a_1^2\zeta_2 - 2224a_1^2\zeta_3 - 4816a_1\zeta_3 + 47724a_1\zeta_4 + \frac{2176a_1^5}{3} + \frac{2752a_1^4}{3} + 12288a_4a_1 \\
& + 18432a_4 - 24576a_5 + 13080\zeta_2\zeta_3 + 29658\zeta_4 - 8922\zeta_5 \Big) \epsilon^5 + \left(24704a_1^4\zeta_2 + 54016a_1^3\zeta_2 \right. \\
& - \frac{8896}{3}a_1^3\zeta_3 - 9632a_1^2\zeta_3 + 255576a_1^2\zeta_4 + 183648a_1\zeta_2\zeta_3 + 301800a_1\zeta_4 - 35688a_1\zeta_5
\end{aligned}$$

$$\begin{aligned}
& +36864a_4\zeta_2 + \frac{95488a_1^6}{45} + \frac{66304a_1^5}{15} + 24576a_4a_1^2 + 73728a_4a_1 - 98304a_5a_1 - 147456a_5 \\
& + 196608a_6 - 49390\zeta_3^2 + 84048\zeta_2\zeta_3 - 41580\zeta_5 + 256873\zeta_6 - 43008s_6 \Big) \epsilon^6 + O(\epsilon^7) \Big\} \quad (\text{A.21})
\end{aligned}$$

$$\begin{aligned}
& \frac{M_{34}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = -\frac{\pi^2(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{2(6\epsilon+1)} \Big\{ 1 + 4a_1\epsilon + (8a_1^2 - 52\zeta_2)\epsilon^2 \\
& + \left(-976a_1\zeta_2 + \frac{32a_1^3}{3} - 70\zeta_3 \right) \epsilon^3 + \left(-5024a_1^2\zeta_2 - 280a_1\zeta_3 + \frac{800a_1^4}{3} + 6144a_4 \right. \\
& \left. - 12637\zeta_4 \right) \epsilon^4 + \left(-\frac{38528}{3}a_1^3\zeta_2 - 560a_1^2\zeta_3 - 167668a_1\zeta_4 + \frac{15488a_1^5}{15} + 24576a_4a_1 \right. \\
& \left. - 13128\zeta_2\zeta_3 - 12234\zeta_5 \right) \epsilon^5 + \left(-\frac{60032}{3}a_1^4\zeta_2 - \frac{2240}{3}a_1^3\zeta_3 - 584936a_1^2\zeta_4 - 269088a_1\zeta_2\zeta_3 \right. \\
& \left. - 48936a_1\zeta_5 + 24576a_4\zeta_2 + \frac{92416a_1^6}{45} + 49152a_4a_1^2 + 101458\zeta_3^2 - 903075\zeta_6 \right. \\
& \left. - 270336s_6 \right) \epsilon^6 + O(\epsilon^7) \Big\} \quad (\text{A.22})
\end{aligned}$$

$$\begin{aligned}
& \frac{M_{35}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = \frac{(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{(2\epsilon+1)(6\epsilon+1)^2} \Big\{ \frac{1}{\epsilon} + 14 + 40\zeta_2\epsilon + (288a_1\zeta_2 \\
& + 272\zeta_2 - 352\zeta_3)\epsilon^2 + (576a_1^2\zeta_2 + 2112a_1\zeta_2 + 64a_1^4 + 1536a_4 - 3584\zeta_3 + 5424\zeta_4)\epsilon^3 \\
& + \left(\frac{256M_{21}^{(1)}}{3} + 1280a_1^3\zeta_2 + 4224a_1^2\zeta_2 + 49632a_1\zeta_4 + \frac{256a_1^5}{5} + 640a_1^4 + 15360a_4 \right. \\
& \left. - 6144a_5 + 18064\zeta_2\zeta_3 + 37824\zeta_4 - 33672\zeta_5 \right) \epsilon^4 + \left(\frac{24064M_{21}^{(1)}}{9} - \frac{M_{35}^{(5)}}{3} + \frac{120320}{3}a_1^3\zeta_2 \right. \\
& \left. - 135680a_1^2\zeta_2 + 411136a_1\zeta_2 + 1555136a_1\zeta_4 + \frac{24064a_1^5}{15} - 9728a_1^4 - 233472a_4 \right. \\
& \left. - 192512a_5 - 367360\zeta_2 + \frac{1698016\zeta_2\zeta_3}{3} - \frac{107776\zeta_3}{3} - 1338816\zeta_4 - 1055056\zeta_5 \right. \\
& \left. + 2177280 \right) \epsilon^5 + O(\epsilon^6) \Big\} \quad (\text{A.23})
\end{aligned}$$

$$\begin{aligned}
& \frac{M_{36}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = -\frac{(4\epsilon-1)(10\epsilon-3)(10\epsilon-1)}{(2\epsilon+1)^2(6\epsilon+1)^2} \Big\{ -\frac{2}{\epsilon} - 36 + (-48\zeta_2 - 176)\epsilon \\
& + (-192a_1\zeta_2 - 544\zeta_2 - 192\zeta_3)\epsilon^2 + (-384a_1^2\zeta_2 - 2304a_1\zeta_2 + 64a_1^4 + 1536a_4 \\
& - 768\zeta_2 - 2336\zeta_3 - 5528\zeta_4)\epsilon^3 + (-4608a_1^2\zeta_2 - 3840a_1\zeta_2 - 15840a_1\zeta_4 + 1024a_1^4 \\
& + 24576a_4 - 9248\zeta_2\zeta_3 - 7104\zeta_3 - 75632\zeta_4 - 24504\zeta_5)\epsilon^4 + \left(-768a_1^4\zeta_2 + 2048a_1^3\zeta_2 \right.
\end{aligned}$$

$$\begin{aligned}
& -7680a_1^2\zeta_2+11520a_1^2\zeta_4-29952a_1\zeta_2\zeta_3-230400a_1\zeta_4-18432a_4\zeta_2+3840a_1^4+92160a_4 \\
& -13328\zeta_3^2-130944\zeta_2\zeta_3-231744\zeta_4-325824\zeta_5-\frac{1285912\zeta_6}{3}-67584s_6\Big)\epsilon^5 \\
& +O(\epsilon^6)\Big\} \quad (\text{A.24})
\end{aligned}$$

$$\begin{aligned}
& \frac{M_{41}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = -\frac{3}{\epsilon} + 9\zeta_2 + 40 + \left(3M_{21}^{(1)}/2 - 180a_1\zeta_2 + 162\zeta_2 + 378\zeta_3 - 88\right)\epsilon \\
& + (14M_{21}^{(1)} + 3M_{22}^{(2)}/2 - 360a_1^2\zeta_2 + 2472a_1\zeta_2 + 72a_1^4 + 1728a_4 - 5992\zeta_2 - 1848\zeta_3 + 3069\zeta_4 \\
& + 1616)\epsilon^2 + M_{41}^{(3)}\epsilon^3 + M_{41}^{(4)}\epsilon^4 + O(\epsilon^5) \quad (\text{A.25})
\end{aligned}$$

$$\begin{aligned}
& \frac{M_{42}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = -\frac{3}{\epsilon} + 40 + (396\zeta_2 - 405\zeta_4 + 368)\epsilon + (-24M_{21}^{(1)} + 1728a_1\zeta_2 \\
& + 1620a_1\zeta_4 - 4632\zeta_2 - 1638\zeta_2\zeta_3 + 3204\zeta_3 + 3240\zeta_4 - 2790\zeta_5 - 12432)\epsilon^2 + (-120M_{21}^{(1)} \\
& - 12M_{21}^{(2)} - 24M_{22}^{(2)} - 3M_{35}^{(5)}/64 - 420a_1^4\zeta_2 + 4128a_1^3\zeta_2 - 15624a_1^2\zeta_2 + 38520a_1\zeta_2 \\
& - 4032a_4\zeta_2 - 17631a_1\zeta_2\zeta_3 - 22779a_1^2\zeta_4 + 145656a_1\zeta_4 - \frac{24a_1^6}{5} + \frac{768a_1^5}{5} - 1368a_1^4 \\
& - 32832a_4 - 18432a_5 - 3456a_6 + 6777\zeta_3^2/4 - 77436\zeta_2 + 73254\zeta_2\zeta_3 - 77436\zeta_3 \\
& - 149373\zeta_4 - 78588\zeta_5 - 3933495\zeta_6/32 + 15480s_6 + 459588)\epsilon^3 + O(\epsilon^4) \quad (\text{A.26})
\end{aligned}$$

$$\begin{aligned}
& \frac{M_{43}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = \frac{48\zeta_2}{5} - \frac{156}{5} + \left(\frac{288a_1\zeta_2}{5} - \frac{3588\zeta_2}{25} + \frac{408\zeta_3}{5} + \frac{5976}{25}\right)\epsilon + \left(\frac{576}{5}a_1^2\zeta_2 \right. \\
& - \frac{12528a_1\zeta_2}{25} - \frac{192a_1^4}{5} - \frac{4608a_4}{5} - \frac{416416\zeta_2}{375} - \frac{35448\zeta_3}{25} + \frac{5916\zeta_4}{5} + \frac{1401232}{375}\Big)\epsilon^2 \\
& + \left(-\frac{768}{5}a_1^3\zeta_2 - \frac{25056}{25}a_1^2\zeta_2 - \frac{1192832a_1\zeta_2}{125} + \frac{7776a_1\zeta_4}{5} + \frac{768a_1^5}{25} + \frac{8352a_1^4}{25} + \frac{200448a_4}{25} \right. \\
& - \frac{18432a_5}{5} + \frac{51526496\zeta_2}{1875} - 816\zeta_2\zeta_3 + \frac{740864\zeta_3}{375} - \frac{355596\zeta_4}{25} + \frac{39288\zeta_5}{5} - \frac{26162464}{625}\Big)\epsilon^3 \\
& + \left(\frac{33408}{25}a_1^3\zeta_2 - \frac{2385664}{125}a_1^2\zeta_2 - \frac{4032}{5}a_1^2\zeta_4 + \frac{78916992a_1\zeta_2}{625} - \frac{12096}{5}a_1\zeta_2\zeta_3 - \frac{338256a_1\zeta_4}{25} \right. \\
& - \frac{18432a_4\zeta_2}{5} - \frac{512a_1^6}{25} - \frac{33408a_1^5}{125} + \frac{1003264a_1^4}{375} + \frac{8026112a_4}{125} + \frac{801792a_5}{25} - \frac{73728a_6}{5} \\
& - \frac{15792\zeta_3^2}{5} - \frac{988918976\zeta_2}{9375} + \frac{43176\zeta_2\zeta_3}{5} + \frac{184730816\zeta_3}{1875} - \frac{28598272\zeta_4}{375} - \frac{2507028\zeta_5}{25} \\
& \left. + \frac{217402\zeta_6}{5} + \frac{50688s_6}{5} + \frac{849324352}{9375}\right)\epsilon^4 + O(\epsilon^5) \quad (\text{A.27})
\end{aligned}$$

$$\begin{aligned}
\frac{M_{44}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= -\frac{6}{\epsilon} + 92 + (-192\zeta_2 - 120)\epsilon + (3712\zeta_2 - 576a_1\zeta_2 - 1104\zeta_3 \\
&- 4432)\epsilon^2 + (16M_{21}^{(1)} - 1152a_1^2\zeta_2 + 12288a_1\zeta_2 + 96a_1^4 + 2304a_4 - 21760\zeta_2 + 14720\zeta_3 \\
&- 17352\zeta_4 + 33312)\epsilon^3 + (576M_{21}^{(1)} + 16M_{22}^{(2)} - 768a_1^3\zeta_2 + 24576a_1^2\zeta_2 - 96768a_1\zeta_2 \\
&- 36432a_1\zeta_4 + 384a_1^5/5 - 512a_1^4 - 12288a_4 - 9216a_5 + 10752\zeta_2 - 21504\zeta_2\zeta_3 + 55936\zeta_3 \\
&+ 326976\zeta_4 - 100476\zeta_5 - 62784)\epsilon^4 + O(\epsilon^5) \quad (\text{A.28})
\end{aligned}$$

$$\begin{aligned}
\frac{M_{45}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= -\frac{6}{\epsilon} + 104 + (224 - 432\zeta_2)\epsilon + (-5184a_1\zeta_2 + 10944\zeta_2 + 8352\zeta_3 \\
&- 22848)\epsilon^2 + (-10368a_1^2\zeta_2 + 112896a_1\zeta_2 - 2304a_1^4 - 55296a_4 - 121344\zeta_2 - 151680\zeta_3 \\
&- 56016\zeta_4 + 358272)\epsilon^3 + (-2048M_{21}^{(1)} - 32256a_1^3\zeta_2 + 225792a_1^2\zeta_2 - 933888a_1\zeta_2 \\
&- 875520a_1\zeta_4 - 3072a_1^5 + 43008a_1^4 + 1032192a_4 + 368640a_5 + 935936\zeta_2 - 372384\zeta_2\zeta_3 \\
&+ 643584\zeta_3 + 1461696\zeta_4 + 887328\zeta_5 - 3808512)\epsilon^4 + \left(-13312M_{21}^{(1)} + 512M_{21}^{(2)} \right. \\
&+ 10M_{35}^{(5)} + 27648a_1^4\zeta_2 - 235520a_1^3\zeta_2 + 2202624a_1^2\zeta_2 - 7836672a_1\zeta_2 + 884736a_4\zeta_2 \\
&+ 921600a_1\zeta_2\zeta_3 + 10368a_1^2\zeta_4 - 15579648a_1\zeta_4 - \frac{13312a_1^6}{5} + \frac{126976a_1^5}{5} + 78848a_1^4 \\
&+ 1892352a_4 - 3047424a_5 - 1916928a_6 - 846240\zeta_3^2 + 4563456\zeta_2 - 4577920\zeta_2\zeta_3 \\
&\left. + 5038592\zeta_3 + 23521920\zeta_4 + 4482048\zeta_5 + 9846480\zeta_6 + 1483776s_6 - 31555584 \right)\epsilon^5 \\
&+ O(\epsilon^6) \quad (\text{A.29})
\end{aligned}$$

$$\begin{aligned}
\frac{M_{51}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= -\frac{6}{\epsilon} + \left(\frac{128\zeta_2}{5} - \frac{6}{5} \right) + \left(768a_1\zeta_2 - \frac{49744\zeta_2}{75} - 448\zeta_3 + \frac{78428}{75} \right)\epsilon \\
&+ \left(1536a_1^2\zeta_2 - \frac{72896a_1\zeta_2}{5} + \frac{3584a_1^4}{5} + \frac{86016a_4}{5} + \frac{442688\zeta_2}{125} + \frac{18880\zeta_3}{3} - 9568\zeta_4 \right. \\
&+ \left. \frac{624232}{375} \right)\epsilon^2 + \left(\frac{38912}{5}a_1^3\zeta_2 - \frac{145792}{5}a_1^2\zeta_2 + \frac{1792256a_1\zeta_2}{25} - 28416a_1\zeta_4 + \frac{8192a_1^5}{5} \right. \\
&- \frac{958912a_1^4}{75} - \frac{7671296a_4}{25} - 196608a_5 + \frac{21599872\zeta_2}{625} + \frac{37248\zeta_2\zeta_3}{5} + 5120\zeta_3 + \frac{2726024\zeta_4}{15} \\
&\left. + 85056\zeta_5 - \frac{282285392}{1875} \right)\epsilon^3 + \left(\frac{57344M_{21}^{(1)}}{15} - \frac{4096M_{21}^{(2)}}{15} - \frac{16M_{35}^{(5)}}{15} + \frac{90112}{5}a_1^4\zeta_2 \right. \\
&- \frac{1180672}{25}a_1^3\zeta_2 - \frac{7269888}{25}a_1^2\zeta_2 + \frac{183588864a_1\zeta_2}{125} - \frac{49152a_4\zeta_2}{5} - \frac{428544}{5}a_1\zeta_2\zeta_3 \\
&- \frac{1314816}{5}a_1^2\zeta_4 + \frac{22281184a_1\zeta_4}{5} + \frac{57344a_1^6}{25} - \frac{667648a_1^5}{25} + \frac{13065472a_1^4}{375} + \frac{104523776a_4}{125} \\
&\left. + \frac{16023552a_5}{5} + \frac{8257536a_6}{5} + \frac{453248\zeta_3^2}{5} - \frac{15643997696\zeta_2}{9375} + \frac{93949856\zeta_2\zeta_3}{75} - \frac{7967104\zeta_3}{15} \right)
\end{aligned}$$

$$-\frac{141720768\zeta_4}{25}-\frac{20465288\zeta_5}{5}-\frac{43744336\zeta_6}{15}-\frac{172032s_6}{5}+\frac{80950072352}{9375}\bigg)\epsilon^4+O(\epsilon^5) \quad (\text{A.30})$$

$$\begin{aligned} \frac{M_{61}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= \frac{108}{5\epsilon} - \frac{224\zeta_2}{5} - \frac{5564}{25} + \left(\frac{168596\zeta_2}{75} - \frac{1344}{5}a_1\zeta_2 - \frac{112\zeta_3}{5} \right. \\ &\quad \left. - \frac{134296}{75} \right) \epsilon + \left(-\frac{2688}{5}a_1^2\zeta_2 + \frac{279952a_1\zeta_2}{25} + \frac{448a_1^4}{5} + \frac{10752a_4}{5} - \frac{3532096\zeta_2}{225} \right. \\ &\quad \left. - \frac{514472\zeta_3}{75} - \frac{32312\zeta_4}{5} + \frac{59912272}{1875} \right) \epsilon^2 + \left(\frac{15712M_{21}^{(1)}}{225} + \frac{559904}{25}a_1^2\zeta_2 - \frac{5176832a_1\zeta_2}{75} \right. \\ &\quad \left. - 30240a_1\zeta_4 + \frac{53024a_1^4}{25} + \frac{1272576a_4}{25} - \frac{792795808\zeta_2}{5625} + \frac{58912\zeta_2\zeta_3}{5} + \frac{132903872\zeta_3}{1125} \right. \\ &\quad \left. + \frac{19776236\zeta_4}{75} - 45304\zeta_5 - \frac{6109675552}{28125} \right) \epsilon^3 + \left(\frac{13531904M_{21}^{(1)}}{3375} - \frac{1792M_{21}^{(2)}}{15} + \frac{15712M_{22}^{(2)}}{225} \right. \\ &\quad \left. - \frac{7M_{35}^{(5)}}{15} + \frac{60928}{15}a_1^4\zeta_2 + \frac{2198144}{25}a_1^3\zeta_2 - \frac{24600064}{75}a_1^2\zeta_2 - \frac{178434176a_1\zeta_2}{1875} + 157696a_4\zeta_2 \right. \\ &\quad \left. - \frac{237888}{5}a_1\zeta_2\zeta_3 - 384832a_1^2\zeta_4 + \frac{17562416a_1\zeta_4}{5} + \frac{35456a_1^5}{15} - \frac{60889856a_1^4}{1125} - \frac{487118848a_4}{375} \right. \\ &\quad \left. - 283648a_5 + 80640\zeta_3^2 + \frac{31107200576\zeta_2}{28125} + \frac{24326024\zeta_2\zeta_3}{25} - \frac{182462144\zeta_3}{375} \right. \\ &\quad \left. - \frac{4725215776\zeta_4}{1125} - \frac{35436356\zeta_5}{25} - \frac{4075022\zeta_6}{3} + \frac{39424s_6}{5} + \frac{1995434100544}{421875} \right) \epsilon^4 \\ &\quad + O(\epsilon^5) \quad (\text{A.31}) \end{aligned}$$

$$\begin{aligned} \frac{M_{62}(3-2\epsilon)}{M_{01}(3-2\epsilon)} &= \frac{192}{5\epsilon} + \frac{1344\zeta_2}{5} - \frac{28896}{25} + \left(\frac{165232\zeta_2}{25} - \frac{56448}{5}a_1\zeta_2 + \frac{81312\zeta_3}{5} \right. \\ &\quad \left. - \frac{362192}{25} \right) \epsilon + \left(\frac{392M_{21}^{(1)}}{5} - \frac{112896}{5}a_1^2\zeta_2 + 295104a_1\zeta_2 - \frac{91392a_1^4}{5} - \frac{2193408a_4}{5} \right. \\ &\quad \left. - \frac{111266224\zeta_2}{375} - \frac{10550208\zeta_3}{25} + \frac{1340304\zeta_4}{5} + \frac{1237871728}{1875} \right) \epsilon^2 + \left(-\frac{114352M_{21}^{(1)}}{75} \right. \\ &\quad \left. + \frac{392M_{22}^{(2)}}{5} - \frac{881664}{5}a_1^3\zeta_2 + 590208a_1^2\zeta_2 - \frac{413288512a_1\zeta_2}{125} + \frac{2346624a_1\zeta_4}{5} - \frac{795648a_1^5}{25} \right. \\ &\quad \left. + \frac{7470464a_1^4}{25} + \frac{179291136a_4}{25} + \frac{19095552a_5}{5} + \frac{18417516896\zeta_2}{5625} - 581952\zeta_2\zeta_3 \right. \\ &\quad \left. + \frac{1384628608\zeta_3}{375} - \frac{66637664\zeta_4}{25} - \frac{3297504\zeta_5}{5} - \frac{722104920704}{84375} \right) \epsilon^3 + \left(\frac{43008M_{21}^{(2)}}{5} \right. \\ &\quad \left. - \frac{28827616M_{21}^{(1)}}{375} - \frac{169232M_{22}^{(2)}}{75} + \frac{784M_{41}^{(3)}}{15} + \frac{168M_{35}^{(5)}}{5} - \frac{1562624}{5}a_1^4\zeta_2 \right. \\ &\quad \left. + \frac{5338112}{25}a_1^3\zeta_2 + \frac{850690176}{125}a_1^2\zeta_2 - \frac{39064847488a_1\zeta_2}{1875} + \frac{172032a_4\zeta_2}{5} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{18982656}{5} a_1 \zeta_2 \zeta_3 + 9606912 a_1^2 \zeta_4 - \frac{2701956096 a_1 \zeta_4}{25} - \frac{243712 a_1^6}{5} + \frac{53266432 a_1^5}{125} \\
& - \frac{19590528 a_1^4}{125} - \frac{470172672 a_4}{125} - \frac{1278394368 a_5}{25} - 35094528 a_6 - 1659840 \zeta_3^2 \\
& + \frac{37182753856 \zeta_2}{1875} - \frac{729078848 \zeta_2 \zeta_3}{25} - \frac{2029091008 \zeta_3}{1875} + \frac{13334743136 \zeta_4}{125} \\
& + \frac{1680498624 \zeta_5}{25} + \frac{380409064 \zeta_6}{5} + \frac{5892096 s_6}{5} - \frac{179237261924672}{1265625} \Big) \epsilon^4 + O(\epsilon^5)
\end{aligned} \tag{A.32}$$

$$\begin{aligned}
& \frac{M_{63}(3-2\epsilon)}{M_{01}(3-2\epsilon)} = -\frac{30}{\epsilon} + 48\zeta_2 + \frac{22}{5} + \left(9504 a_1 \zeta_2 - 998 \zeta_2 - 6888 \zeta_3 + \frac{48788}{25} \right) \epsilon \\
& + \left(28 M_{21}^{(1)} + 19008 a_1^2 \zeta_2 - 109032 a_1 \zeta_2 + 11424 a_1^4 + 274176 a_4 - \frac{15192 \zeta_2}{5} + \frac{125692 \zeta_3}{5} \right. \\
& - 141588 \zeta_4 + \frac{105856648}{1125} \Big) \epsilon^2 + \left(\frac{10696 M_{21}^{(1)}}{5} + 28 M_{22}^{(2)} + 116736 a_1^3 \zeta_2 - 218064 a_1^2 \zeta_2 \right. \\
& - \frac{859552 a_1 \zeta_2}{5} - 814320 a_1 \zeta_4 + 24576 a_1^5 - \frac{702736 a_1^4}{5} - \frac{16865664 a_4}{5} - 2949120 a_5 \\
& + \frac{51276928 \zeta_2}{225} + 463536 \zeta_2 \zeta_3 + \frac{18373168 \zeta_3}{25} + \frac{14592814 \zeta_4}{5} + 1144380 \zeta_5 \\
& - \frac{10884323888}{5625} \Big) \epsilon^3 + \left(\frac{7617712 M_{21}^{(1)}}{75} - 4096 M_{21}^{(2)} + \frac{28168 M_{22}^{(2)}}{15} + \frac{56 M_{41}^{(3)}}{3} \right. \\
& - 16 M_{35}^{(5)} + 344704 a_1^4 \zeta_2 - \frac{39488}{5} a_1^3 \zeta_2 - \frac{34743744}{5} a_1^2 \zeta_2 + \frac{578604032 a_1 \zeta_2}{25} \\
& + 1637376 a_4 \zeta_2 + 12672 a_1 \zeta_2 \zeta_3 - 5156736 a_1^2 \zeta_4 + \frac{394249896 a_1 \zeta_4}{5} + \frac{173824 a_1^6}{5} \\
& - \frac{6743232 a_1^5}{25} - \frac{12002624 a_1^4}{25} - \frac{288062976 a_4}{25} + \frac{161837568 a_5}{5} + 25030656 a_6 \\
& + 1519512 \zeta_3^2 - \frac{7949697856 \zeta_2}{375} + 20969628 \zeta_2 \zeta_3 + \frac{1712657728 \zeta_3}{1125} - \frac{6405160952 \zeta_4}{75} \\
& - 65011022 \zeta_5 - 41811552 \zeta_6 - 2042880 s_6 + \frac{3579868183328}{28125} \Big) \epsilon^4 + O(\epsilon^5)
\end{aligned} \tag{A.33}$$

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